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Generalization of the h-Deformation to Higher Dimensions[1]

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Abstract

In this article we construct $GL_h(3)$ from $GL_q(3)$ by a singular map. We show that there exist two singular maps which map $GL_q(3)$ to new quantum groups. We also construct their R-matrices and will show although the maps are singular but their R-matrices are not. Then we generalize these singular maps to the case $GL(N)$ and for C_n series.

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There exist two types of $SL(2)$ quantum groups. One is the standard $SL_q(2)$, another one is the Jordanian quantum group which is also called the h-deformation of $SL(2)$. Quantum matrices in two dimensions, admitting left and right quantum spaces, are classified [1]. One is the q-deformation of $GL(2)$, the other is the h-deformation. The q-deformation of $GL(N)$ has been studied extensively but in the literature only the two dimensional case of h-deformation has been studied.[2-7]

In ref. 8 it is shown that $GL_h(2)$ can be obtained from $GL_q(2)$ by a singular limit of a similarity transformation. We will show this method can be used successfully, for construction of $GL_h(N)$. In other words, at first we will consider the $GL(3)$, and introduce two singular maps which convert $GL_q(3)$ to $GL_h(3)$. Then we generalize one of the singular maps to N-dimensional case. We will use R-matrix of $GL_q(N)$ which by this map, results to a new R-matrix. Also, by this map one can obtain h-deformation of C_n series, but can not for B_n and D_n series.

In this article we denote q-deformed objects by primed quantities. Unprimed quantities represent transformed objects.

Consider Manin's q-plane with the following quadratic relation between coordinates.

$$x'y' = qy'x'. \tag{1}$$

By the following linear transformation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \frac{h}{q-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

the relation (1) changes to $xy - qyx = hy^2$. For the case of $q = 1$, one get the relation of two dimensional h-plane. In fact g itself is singular in the $q = 1$ case, but the resulting relation for the plane is non-singular.

The above linear transformation on the plane induces the following similarity transformation on the R-matrix of $GL_q(2)$.

$$R_h = \lim_{q\to 1} (g \otimes g)^{-1} R_q (g \otimes g). \tag{3}$$

Although the above map is singular, the resulting R-matrix is non-singular and is the well known R-matrix of $GL_h(2)$.

Now consider 3-dimensional Manin's quantum space:

$$x'_i x'_j = q x'_j x'_i \quad i < j, \tag{4}$$

and consider the following linear transformation:

$$X = g^{-1} X', \tag{5}$$

where

$$g = \begin{pmatrix} \lambda_1 & \alpha & \beta \\ 0 & \lambda_2 & \gamma \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{6}$$

Here α, β and γ are parameters which can be singular at $q = 1$. So they can be written as $\frac{1}{f(q)}$ where $f(1) = 0$. The Taylor expansion of $f(q)$ about $q = 1$ is $f(q) = \frac{1}{h}(q-1) + O((q-1)^2)$. We need only the first term, because we are only interested in the behaviour of $f(q)$ in the neighbourhood of $q = 1$. The coefficient of first term in the Taylor expansion, h, plays the role of the deformation parameter for the new quantum group. The λ_is can be made equal to 1 by rescaling.

To obtain α, β and γ we should apply this map to the q-deformed plane and its dual, and require that the mapped plane and its dual be non-singular at $q = 1$. The following are the only singular maps satisfying this condition:

$$g_1 = \begin{pmatrix} 1 & \frac{h}{q-1} & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \frac{h}{q-1} \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & \alpha & \frac{h}{q-1} \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

Here $\alpha,\ \beta$ and γ (in $g_1,\ g_2,\ g_3,$) are non-singular parameters. Note that the R-matrices obtained from these maps, solve the quantum Yang-Baxter equation and are non-singular for $q = 1$.

Let us denote the dependence of g_1, g_2 and g_3 on parameters explicitly:

$$g_1 := g_1(\tfrac{h}{(q-1)}, \beta), \quad g_2 := g_2(\tfrac{h}{(q-1)}, \alpha, \beta), \quad g_3 := g_3(\tfrac{h}{(q-1)}, \alpha, \gamma). \tag{8}$$

It is easy to show that:

$$g_1(\frac{h}{(q-1)}, \beta) g_1(0, -\beta) \quad = \quad g_1(\frac{h}{(q-1)}, 0)$$

$$
\begin{aligned}
g_2(\frac{h}{(q-1)},\alpha,\beta)g_2(0,-\alpha,-\beta) &= g_2(\frac{h}{(q-1)},0,0) \\
g_3(\frac{h}{(q-1)},\alpha,\gamma)g_3(\alpha\gamma,-\alpha,-\gamma) &= g_3(\frac{h}{(q-1)},0,0).
\end{aligned}
\tag{9}
$$

so all non-singular parameters in the above matrices can be set to zero. Moreover the R-matrices $R(g_1)$ and $R(g_2)$ which are obtained by formula (3) using $g_1(\frac{h}{(q-1)},0)$ and $g_2(\frac{h}{(q-1)},0,0)$ respectively, are equivalent, because:

$$
(s\otimes s)^{-1}R(g_2)(s\otimes s)=R(g_1). \tag{10}
$$

where $s=e_{13}+e_{21}+e_{32}$. So, there are only two independent cases. The R-matrices corresponding to these transformations are non-singular and have been first obtained by Hietarinta [9]. The first case (the trivial case) is $\beta=0$ in g_1 (or $\alpha=\beta=0$ in g_2) and the second case is $\alpha=\gamma=0$ in g_3. The h-deformed quantum plane and its dual and R-matrices corrsponding to these cases are:

First case

$$
\begin{aligned}
&[x_1,x_2]=hx_2^2, && \eta_3^2=\eta_2^2=\{\eta_1,\eta_2\}=0,\\
&[x_1,x_3]=0, && \{\eta_2,\eta_3\}=\{\eta_1,\eta_3\}=0,\\
&[x_2,x_3]=0, && \eta_1^2=-h\eta_2\eta_1.
\end{aligned}
\tag{11}
$$

and the non-zero elements of R-matrix except for $R_{ijij}=1$ are:

$$
\begin{aligned}
R_{1121}=R_{2122} &= -R_{1112}=-R_{1222}=h,\\
R_{1122} &= h^2.
\end{aligned}
\tag{12}
$$

Second case

$$
\begin{aligned}
&[x_1,x_2]=2hx_3x_2, && \{\eta_1,\eta_2\}=-2h\eta_3\eta_2,\\
&[x_1,x_3]=hx_3^2, && \eta_1^2=-h\eta_3\eta_1,\\
&[x_2,x_3]=0, && \eta_3^2=\eta_2^2=\{\eta_1,\eta_3\}=\{\eta_2,\eta_3\}=0.
\end{aligned}
\tag{13}
$$

and the non-zero elements of R-matrix except for $R_{ijij}=1$ are:

$$
\begin{aligned}
&R_{1113}=R_{1333}=-h, \quad R_{1131}=R_{3133}=h,\\
&R_{2132}=-R_{1223}=2h \quad R_{1133}=h^2.
\end{aligned}
\tag{14}
$$

A linear transformation on the plane induces a similarity transformation on the quantum matrices acting upon it.

$$M' = gMg^{-1}, \tag{15}$$

The algebra of functions, $GL_q(3)$, is obtained from the following relations:

$$R'M'_1M'_2 = M'_2M'_1R'. \tag{16}$$

Applying transformation (15) one easily obtains for the case of $q = 1$.

$$RM_1M_2 = M_2M_1R. \tag{17}$$

So the entries of the transformed quantum matrix M fulfill the commutation relations of the $GL_h(3)$, for both g's. It is easy to show that the h-deformed determinant is central, so it can be set to 1. A quantum group's differential structure is completely determined by R-matrix [10]. One therefore expects that by these similarity transformations the differential structure of the h-deformaton be obtained from that of the q-deformation.

$$\begin{aligned} M_2dM_1 - R_{12}dM_1M_2R_{21} &= 0, \\ dM_2dM_1 + R_{12}dM_1dM_2R_{21} &= 0. \end{aligned} \tag{18}$$

Now, it is obvious that, defining $dM := g^{-1}dMg$ and using the above relations the differential of $GL_h(3)$ can be easily obtained from the corresponding differential structure of $GL_q(3)$.

For the higher dimensions, there are several generalizations which depend on the position of singularity in g. For example we consider the following generalization:

$$g = \sum_{i=1}^{N} e_{ii} + \frac{h}{q-1} e_{1N} \tag{19}$$

The general aspect of the contraction for arbitrary N can be obtained from this simple map. By inserting this map in (3) we will obtain the general form of the h-deformed R-matrix, which solves the quantum Yang-Baxter equation.

1- The series A_{n-1}

After applying this singular map, the corresponding h-deformed R-matrix will become:

$$\begin{aligned} R_h &= \sum_{i,j=1}^{N} e_{ii} \otimes e_{jj} + 2h \sum_{i>1}^{N-1} (e_{1i} \otimes e_{iN} - e_{iN} \otimes e_{1i}) \\ &- h(e_{1N} \otimes e_{NN} - e_{NN} \otimes e_{1N}) - h(e_{11} \otimes e_{1N} - e_{1N} \otimes e_{11}) \\ &+ h^2(e_{1N} \otimes e_{1N}). \end{aligned} \tag{20}$$

Consider N-dimensional q-deformed quantum space

$$x'_i x'_j = q x'_j x'_i \quad i < j. \tag{21}$$

Assume the following linear singular transformation

$$x'_i = g_{ij} x_j. \tag{22}$$

By the above transformation and in the $q = 1$ case we obtain the h-deformed quantum plane as follows:

$$\begin{aligned} &x_i x_j = x_j x_i && 1 < i < j \leq N, \\ &[x_1, x_j] = 2h x_N x_j, && [x_1, x_N] = h(x_N)^2. \end{aligned} \tag{23}$$

2- The series B_n, C_n and D_n

The corresponding q-deformed R-matrix has order $N^2 \times N^2$, where $N = 2n+1$ for B_n and $N = 2n$ for D_n and C_n and it is given by [11]:

$$\begin{aligned} R_q &= q \sum_{i \neq i'}^{N} e_{ii} \otimes e_{ii} + e_{\frac{N+1}{2}\frac{N+1}{2}} \otimes e_{\frac{N+1}{2}\frac{N+1}{2}} + \sum_{i \neq j, j'}^{N} e_{ii} \otimes e_{jj} \\ &+ q^{-1} \sum_{i \neq i'}^{N} e_{i'i'} \otimes e_{ii} + (q - q^{-1}) \sum_{i>j}^{N} e_{ij} \otimes e_{ji} \\ &- (q - q^{-1}) \sum_{i>j}^{N} q^{\rho_i - \rho_j} \epsilon_i \epsilon_j e_{ij} \otimes e_{i'j'}. \end{aligned} \tag{24}$$

The second term is present only for the series B_n. Here $i' = N+1-i, j' = N+1-j, \epsilon_i = 1, i = 1, ..., N$ for the series B_n and D_n, $\epsilon_i = 1, i = 1, ..., \frac{N}{2}$, $\epsilon_i = -1, i = \frac{N}{2} + 1, ..., N$ for the series C_n and $(\rho_1, ..., \rho_N)$ is:

$$\begin{aligned} &(\tfrac{n-1}{2}, ..., \tfrac{1}{2}, 0, \tfrac{1}{2}, ..., -n + \tfrac{1}{2}) && for B_n \\ &(n, n-1, ..., 1, -1, ..., -n) && for C_n \\ &(n-1, ..., 1, 0, 0, -1, ..., -n+1) && for D_n \end{aligned} \tag{25}$$

By inserting this R-matrix in (3), the coefficient of $e_{1N} \otimes e_{1N}$ will become:

$$\frac{h^2}{q-1}(q^{-1}+1)(1+\epsilon_N q^{\rho_N-\rho_1}). \tag{26}$$

This expression is non-singular only when $\epsilon_N = -1$ and for $q = 1$ it is equal to $2Nh^2$. We thus see that only the C_n series remains non-singular. The corresponding h-deformed R-matrix is:

$$\begin{aligned} R_h &= \sum_{i,j=1}^{N} e_{ii} \otimes e_{jj} + 2Nh^2 e_{1N} \otimes e_{1N} \\ &- 2h \sum_{i=2}^{N} e_{1i} \otimes e_{iN} + \epsilon_i e_{iN} \otimes e_{i'N} \\ &+ 2h \sum_{i=1}^{N-1} e_{iN} \otimes e_{1i} - \epsilon_i e_{1i} \otimes e_{1i'}. \end{aligned} \tag{27}$$

So by this method we can obtain $SP_h(2n)$. The algebra $SP_q^{2n}(c)$ with generators $x'_1, ..., x'_{2n}$ and relations

$$R'_q(x' \otimes x') = q x' \otimes x', \tag{28}$$

is called the algebra of functions on quantum $2n$-dimensional symplectic space. After applying the singular transformation (19) to (28) we obtain the relations between the generators of $SP_h^{2n}(c)$:

$$x_i x_j = x_j x_i, \quad 1 < i < j \leq N, \quad j \neq j', \tag{29}$$

$$\begin{aligned} x_1 x_j &= x_j x_1 + 2h x_N x_j, \quad j \neq N, \\ x_{i'} x_i &= x_i x_{i'} + 2h \epsilon_{i'} x_N^2, \quad 1 < i < i' \leq N. \end{aligned} \tag{30}$$

In $SP_q^{2n}(c)$ the equality $x'^t C' x' = 0$ holds. By applying the singular map (29), C' transforms to $C = g^t C' g$, where C is given by:

$$C = \sum_{i=1}^{N} \epsilon_i e_{ii'} - Nh e_{NN}. \tag{31}$$

The Quantum group $SP_q(2n)$ acts on $SP_q^{2n}(c)$ and preserves $x'^t C' x' = 0$, so we have:

$$M'^t C' M' = C', \tag{32}$$

on the other hand:

$$M = gM'g^{-1}, \quad M^t = (g^{-1})^t M'^t g^t. \tag{33}$$

It follows that:

$$M^t C M = C, \tag{34}$$

So we conclude that the quantum group $SP_h(2n)$ acts on $SP_h^{2n}(c)$ and preserves $x^t C x = 0$. It is interesting to note that the expression $x'^t C' x'$, which should be equal to 1 for $SO(2n)$ and $SO(2n+1)$ (B_n and D_n series), is singular. So we cannot obtain the h-deformation of B_n and D_n series by contraction of the q-deformation, at least by this form (upper triangular matrix) of singular transformation (g).

One of the interesting problems is to construct $U_h(gl(3))$, and its generaliztion to higher dimensions.

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